Cable Motion of a Spinning Spring-Mass System in Orbit

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Equations are derived by Hamilton's principle for the in-plane motion of a cable-connected satellite spinning in orbit when both extensible and lateral vibrations are considered. These equations are linearized and solved after neglecting Coriolis coupling for three example cases: 1) nonspinning in free space, 2) spinning in free space, and 3) spinning in orbit. Lagrange's equations are derived for a lumped-mass approximation to the cable system constrained to move only in the plane of the orbit. These nonlinear equations are solved numerically using a fourth-order Runge-Kutta routine. The examples calculated correspond to a single 16,000-ft wire attached between a large central mass moving in an undisturbed Keplerian orbit and a smaller (~100 lb) spinning outer mass point. It is shown that viscous axial damping is ineffective in reducing transverse cable oscillations, that very few mass points are required to give good accuracy, and that computer experiments are feasible.

Nomenclature

A = cross-sectional area of cable
C = damping coefficient
e = spring length
E = Young's modulus of cable
G = universal gravitational constant times mass of earth
J = action integral
k,l = spring constant and length, respectively
$L,\mathfrak{L}$ = Lagrangian and Lagrangian density, respectively
M = mass
Q = generalized force
$\bar{r}$ = radius vector from Earth center to any mass point
R = orbital radius
T = kinetic energy
u,v = displacements of cable along, and perpendicular to its length, respectively
V = potential energy
γ = $2(\phi - \theta)$
ϵ = strain
η = radial coordinate
θ = orbital angle measured clockwise from perigee
ρ = density of cable/unit length
ϕ = relative angle of cable

Subscripts

E,G = elastic and gravity potentials, respectively
i,j = ith or jth mass point
M,s = end mass and string, respectively
$(\dot{\ })$ = time derivative
η = partial derivation with respect to η
$0(\ldots)$ = a quantity of the same order of magnitude as $(\ldots)$

Introduction

INTEREST continues in flexible satellites composed of multiple masses connected by cables. The impetus for the present work arose from a NASA grant to the Radio Astronomy Observatory of the University of Michigan to study the feasibility of a very large, cable-connected, orbiting and spinning antenna[11] for radio astronomy observation at wave lengths near 1 Mc. The single mass model considered in this study is, in fact, an approximation to a portion of that structure. The general approach to the problem was devised to allow analysis of the complete structure in orbit.

The properties of the dumbbell configuration with a massless connecting cable with extensible vibrations permitted have been studied by Paul,[1] Pittman and Hall,[2] Austin,[3,4] Pengelly,[5] Tai and Loh,[6] Chobotov,[7,8] and the authors.[9] When the mass of the cable is included, lateral motion must be considered. Lateral dynamics of the cable has been investigated by Chobotov,[7] Pengelly,[5] Tai and Loh,[6] and Targoff.[10] Each considers a cable-connected system, spinning in the plane of the orbit, and assumes that the cable has negligible bending stiffness and is made of a uniform, linear elastic material. The bodies which the cable connects are assumed to be point masses by all authors except Pengelly, who considers finite rigid bodies as well. Targoff carries the analysis furthest. He derives the equations of motion by summing forces on each mass element of the system and satisfying static equilibrium with the corresponding D'Alembert forces. An assumption of small deflections is made from the start of the analysis. Unfortunately, the general equations derived for the spinning cable system are too complex to lend themselves to analytical solution if axial motion and large deflections are considered.

In the present investigation large deflections are permitted, Coriolis forces are retained, and a numerical method of solution is used. The method is applied to the special case of a point mass spinning about a much heavier body to which it is connected by a linear elastic, constant-cross-sectional-area cable. The system, in turn, is in orbit about a third body as shown in Fig. 1.

The equations of motion are derived by applying Hamilton's principle to the action integral for the cable-connected system shown in Fig. 1. They are nonlinear, coupled, partial differential equations with time dependent boundary conditions. Three special cases are derived from the general equations when appropriate assumptions are imposed. The complexity of the general nonlinear equations and the corresponding linearized equations seems to preclude analytical solution unless some rather severe assumptions are made. Thus, even though a systematic approach to the problem via Hamilton's principle produces the exact equations of motion,

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it appears that a solution cannot be obtained without limiting assumptions.

Because of the foregoing difficulties, the cable is approximated by discrete masses and linear elastic, massless springs. The Lagrange equations of motion for the lumped system are shown to be a system of nonlinear, ordinary differential equations. Damping is included in the equations and its effects on stability investigated.

A justification of the lumped element analysis is shown by comparing natural frequencies obtained from both a distributed mass and a lumped mass analysis of the nonspinning cable. Damping is shown to be ineffective with respect to transverse motions. A brief justification is given for neglecting out-of-plane motions of the connecting cable in any analysis of in-plane motions.

Equations of Motion for Continuous Cable System

The Lagrangian of the cable shown in Fig. 1 is

$$L_s = T_s - V_E - V_G \tag{1}$$

where T_s, the kinetic energy of cable, is given by

$$T_s = \frac{1}{2}\rho \int_0^l |\dot{\bar{r}}|^2 \, d\eta \tag{2}$$

where l is the length of unstretched cable, ρ is its mass/unit length, and $\bar{r}$ is the radius vector from the Earth's center to any mass point dm and is given by

$$\bar{r} = [R \cos\theta + (\eta + v) \cos\phi - u \sin\phi]\bar{\imath} + [R \sin\theta + (\eta + v) \sin\phi + u \cos\phi]\bar{\jmath} \tag{3}$$

The elastic potential, V_E, is given by

$$V_E = \frac{1}{2}\int_0^l AE\epsilon^2 d\eta \tag{4}$$

where ϵ is the axial strain of the cable, A is the cross-sectional area of the cable, and E is Young's modulus. This definition of strain energy assumes only axial strains and neglects bending strains. This assumption is valid for a long thin cable, even when transverse deflections are allowed, if the local radius of curvature remains large. The axial strain is given by

$$\epsilon = (1 + 2v_\eta + v_\eta^2 + u_\eta^2)^{1/2} - 1 \tag{5}$$

where $v_\eta \equiv \partial v/\partial\eta$ and $u_\eta \equiv \partial u/\partial\eta$. For small displacements, this reduces to the familiar one-dimensional strain-displacement relation, $\epsilon = v_\eta$. The potential due to gravity V_G is given by

$$V_G = -\int_0^l \frac{G\rho}{|\bar{r}|} \, d\eta \tag{6}$$

The Lagrangian for the end mass is found by replacing the integral of $\rho d\eta$ by M and evaluating the functions at $\eta = l$ in Eqs. (2) and (6), i.e.,

$$T_M = \tfrac{1}{2}M|\dot{\bar{r}}|^2\Big|_{\eta=l} \tag{7}$$

The action integral is defined as

$$J(v,u) = \int_{t_1}^{t_2} L dt \tag{8}$$

where $L = L_s + L_M$. According to Hamilton's principle the first variation of J must be stationary. After some manipulation we obtain

$$\partial\mathcal{L}/\partial v - \partial(\partial\mathcal{L}/\partial v_\eta)/\partial\eta - \partial(\partial\mathcal{L}/\partial\dot{v})/\partial t = 0 \tag{9}$$

and an equation of identical form in u, the boundary conditions at $\eta = l$ are

$$\partial L/\partial v_\eta\Big|_{\eta=l} + \partial L_M/\partial v - \partial(\partial L_M/\partial\dot{v})/\partial t = 0 \tag{10}$$

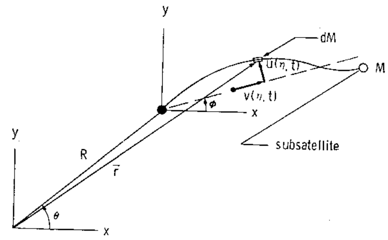

Fig. 1 Coordinate system for distributed mass wire-subsatellite system.

and, again, one of identical form in u. Equations (9) and (10) may be simplified by expanding the terms which come from the gravity potential in terms of the ratio of the body dimensions to the orbital radius. Since $R \gg \eta$, v, u, we may keep up to first-order terms and obtain the following set of nonlinear, partial differential equations for plane motion for the linear elastic sub-satellite-cable system. Letting $\gamma \equiv 2(\phi - \theta)$, s = sin, c $\equiv$ cos, and $\zeta \equiv (1 + 2v_\eta + v_\eta^2 + u_\eta^2)^{-1/2}$,

$$\ddot{v} = (\eta + v)[\dot\phi^2 + (1 + 3c\gamma)G/2R^3] + u[\ddot\phi - s\gamma(3G/2R^3)] + 2\dot{u}\dot\phi + (AE/\rho)[v_{\eta\eta}(1 - \zeta) + (1 + v_\eta)(v_{\eta\eta} + v_\eta v_{\eta\eta} + u_\eta u_{\eta\eta})\zeta^3] - (R\ddot\theta + 2\dot{R}\dot\theta)s\gamma/2 - (\ddot{R} - R\dot\theta^2 + G/R^2)c\gamma/2 \tag{11}$$

$$\ddot{u} = u[\dot\phi^2 + (1 - 3c\gamma)G/2R^3] - (\eta + v)[\ddot\phi - s\gamma(3G/2R^3)] - 2\dot{v}\dot\phi + (AE/\rho)[u_{\eta\eta}(1 - \zeta) + u_\eta(v_{\eta\eta} + v_\eta v_{\eta\eta} + u_\eta u_{\eta\eta})\zeta^3] - (\ddot{R} - R\dot\theta^2 + G/R^2)s\gamma/2 - (R\ddot\theta + 2\dot{R}\dot\theta)c\gamma/2 \tag{12}$$

If R and θ are governed by the Keplerian equations of motion, then $\ddot{R} - R\dot\theta^2 + G/R^2 = 0$ and $R\ddot\theta + 2\dot{R}\dot\theta = 0$, so that the last two terms in each of Eqs. (11) and (12) are zero.

If the boundary conditions of Eqs. (11) and (12) are treated accordingly, then at $\eta = l$

$$\ddot{v} = (\eta + v)\{\dot\phi^2 + (G/2R^3)[1 + 3\,c\gamma]\} + u\{\ddot\phi - (3G/2R^3)s\gamma\} + 2\dot{u}\dot\phi - (AE/M)[1 - \zeta](1 + v_\eta) \tag{13}$$

$$\ddot{u} = u\{\dot\phi^2 + (G/2R^3)[1 - 3c\gamma]\} - (\eta + v)[\ddot\phi - (3G/2R^3)s\gamma] - 2\dot{v}\dot\phi - (AE/M)[1 - \zeta]u_\eta$$

The equations of motion have two independent variables, time and distance. They are forced through the gyroscopic and gravity-gradient terms and coupled through the gyroscopic, the elastic, and the gravity-gradient terms. The boundary conditions for the free end exhibit the same form as the equations of motion and are time dependent. An analytic solution of this set of equations is not feasible. However, there are certain interesting and informative special cases that may be studied if simplifying assumptions are made.

Case I. Nonspinning in Free Space

If we let $R \to \infty$, $\dot\phi = 0$, and $AEv_\eta = T$ = constant, the equations degenerate to the familiar special case of a cable vibrating with constant tension between fixed walls. The uncoupled linearized equations which result are both wave equations with the natural frequencies

$$\omega_n = (n\pi/l)(T/\rho)^{1/2} \qquad n = 1,2,\ldots \tag{14}$$

and

$$\omega_n = (n\pi/l)(AE/\rho)^{1/2} \qquad n = 1,3,5,\ldots$$

in the lateral and axial directions respectively.

Case II. Spinning in Free Space

If we let $R \to \infty$, $\dot\phi = \omega$ = const and let the tension be constant with respect to time but vary along the length of the cable, we eliminate the gravity effects. When the radial displacement is a function of η alone, it implies that the end mass is allowed to seek a steady-state equilibrium position along the η,ϕ reference line and then is held fixed relative to that line for the remainder of the problem. This has the effect of eliminating the time-dependent boundary conditions, since the position of the end mass is fixed in the rotating reference frame.

The following equations result:

$$v_{\eta\eta} + (\rho\omega^2/AE)v = -(\rho\omega^2/AE)\eta \tag{15}$$

and

$$\ddot{u} = u\omega^2 + (AE/\rho)(u_\eta v_{\eta\eta} + v_\eta u_{\eta\eta})$$

with boundary conditions at $n = 0$

$$u = v \equiv 0 \tag{16}$$

and at $\eta = l$

$$u = 0 \qquad (l + v)\omega^2 - (AE/M)v_\eta = 0 \tag{17}$$

In these equations the Coriolis coupling terms have been neglected, even though this is of doubtful validity, in order to obtain analytical solutions. To obtain solutions we note that

$$v(\eta) = B_1 \sin\alpha\eta + B_2 \cos\alpha\eta - \eta \tag{18}$$

where $\alpha = (\rho\omega^2/AE)^{1/2}$, and if $v \ll l$, the boundary conditions on v for $t \geq 0$ become

$$v(0,t) = 0 \quad AEv_\eta(l,t) = M\omega^2 l \tag{19}$$

and thus

$$B_2 = 0 \quad \alpha B_1 = (1 + M\omega^2 l/AE)/\cos\alpha l \tag{20}$$

The remaining equation then reduces to

$$\ddot{u} = u\omega^2 + (AE/\rho)[u_{\eta\eta}(\alpha B_1 \cos\alpha\eta - 1) - u_\eta \alpha^2 B_1 \sin\alpha\eta] \tag{21}$$

For $\alpha \ll 1$, i.e., small spin rates, we get

$$\alpha B_1 \cos\alpha\eta = (\cos\alpha\eta/\cos\alpha l)[1 + M\omega^2 l/AE] \cong [1 + \alpha^2(l^2 - \eta^2)/2] \; [1 + M\omega^2 l/AE] \tag{22}$$

Hence Eq. (21) reduces to

$$\ddot{u} = u\omega^2 + \{u_{\eta\eta}[(l^2 - \eta^2) + Ml/\rho] - u_\eta \eta\}\omega^2 \tag{23}$$

Letting

$$u(\eta,t) = X(\eta)T(t) \tag{24}$$

yields

$$\frac{\ddot{T}}{T} = -\beta^2 = \omega^2 + \frac{X_{\eta\eta}}{X}\left[-\frac{\eta^2}{2} + \frac{l^2}{2} + \frac{Ml}{\rho}\right]\omega^2 - \frac{X_\eta}{X}\eta\omega^2 \tag{25}$$

and

$$X_{\eta\eta}(C - \eta^2/2)\omega^2 - \eta\omega^2 X_\eta + (\beta^2 + \omega^2)X = 0 \tag{26}$$

where $C \equiv l^2[\frac{1}{2} + M/\rho l]$. If C is large, such as it would be for a large tip mass to cable weight ratio, and if we divide by C and neglect terms of order η/C, and define $\alpha_1 \equiv [(\beta^2 + \omega^2)/c\omega^2]^{1/2}$ Eq. (26) reduces to

$$X_{\eta\eta} + \alpha_1^2 X = 0 \tag{27}$$

This is the vibrating string equation again, modified by the rotational rate ω^2. Hence

$$X(\eta) = B_1 \sin\alpha_1\eta + B_2 \cos\alpha_1\eta \tag{28}$$

The boundary conditions are $X(0) = 0$ and $X(l) = 0$ which gives

$$B_2 = 0; \quad \text{and } \sin\alpha_1 l = n\pi \tag{29}$$

$$\beta^2 = \omega^2 Cn^2\pi^2/l^2 - \omega^2 = \omega^2[(\eta\pi)^2 M/\rho l - 1]$$

This shows that the frequency increases as the spin rate increases, the tip mass increases, or cable mass ρl decreases.

The effects of spinning and the effects of the corresponding tension increase can be separated as follows: from Eq. (15)

$$\ddot{u} = u\omega^2 + (T/\rho)u_{\eta\eta} \tag{30}$$

for a constant tension which is not necessarily the steady-state equilibrium tension

$$X_{\eta\eta} + (\rho/T)(\omega^2 + \beta^2)X = 0 \tag{31}$$

and

$$X(\eta) = B_1 \sin\alpha_3\eta \quad \text{where} \quad \alpha_3 = [(\rho/T)(\omega^2 + \beta^2)]^{1/2} \tag{32}$$

which gives

$$\beta^2 = T(n\pi)^2/\rho l^2 - \omega^2 \qquad n = 1,2,\ldots \tag{33}$$

This implies that for constant prescribed tension, spinning the system reduces the natural frequencies. Now we substitute for T, the steady-state equilibrium values, and we get

$$\beta^2 = \omega^2[(M/\rho l)(n\pi)^2 - 1] \qquad n = 1,2,\ldots \tag{34}$$

which shows that spinning increases the natural frequencies only if the tension is determined by the steady-state equilibrium position of the end mass.

Case III. Spinning in Orbit

If we now let R be governed by the Keplerian equations of motion, $\dot{\phi}$ be governed by the equations for a rigid rod spinning in orbit, the tension again be only a function of η, and neglect the Coriolis coupling, we can again separate variables and obtain solutions. As pointed out for Case II, neglect of the Coriolis coupling cannot be justified and may be a severe limitation on the results obtained by this analysis but is necessary to achieve a solution.

The steady-state tension AEv is governed by the following:

$$(\rho/AE)(\eta + v)[\dot{\phi}^2 + (1 + 3c\gamma)G/2R^3] + v_{\eta\eta} = 0 \tag{35}$$

with boundary conditions

$$AEv_\eta(l) = Ml[\dot{\phi}^2 + (1 + 3c\gamma)G/2R^3] \tag{36}$$

The expression for $v_\eta(\eta)$ which results is

$$v_\eta(\eta) = [AEMl + AE\rho(l^2 - \eta^2)] \times [\dot{\phi}^2 + (1 + 3c\gamma)G/2R^3] \tag{37}$$

This and the expression for $v_{\eta\eta}(\eta)$ is substituted into

$$\ddot{u} = u[\dot{\phi}^2 + (1 - 3c\gamma)G/2R^3] - \eta[\ddot{\phi} - (3G/R^3)s\gamma] \; [u_{\eta\eta}(l^2 - \eta^2 + Ml/\rho) - u_\eta\eta][\dot{\phi}^2 - (G/2R^3)(3s\gamma)] \tag{38}$$

Equation (38) is the same as Eq. (23) if $R = \infty$, and $\dot{\phi} = \omega$, and $\ddot{\phi} = 0$. The mode shapes can be obtained for certain simplified cases as shown for the free-space solution of Eq. (23).

The corresponding time varying equation, after separating variables, is of the form

$$\ddot{T} = [\delta + \epsilon f(t)]T = 0 \tag{39}$$

The details of the separation of variables will not be shown here, since only the general form of the time-dependent function $T(t)$ is discussed. In order that separation of variables be applicable, we must have

$$\ddot{\phi} = (3G/R^3)s\gamma = \dot{\phi}d\dot{\phi}/d\phi \tag{40}$$

or

$$(\dot{\phi})^2 = -(3G/2R^3)c\gamma + [\dot{\phi}(0)]^2$$

This differential equation governs the variation of ϕ for the reference frame and introduces the time dependency $f(t)$ into Eq. (39). The preceding equation for ϕ is the equation for a rigid dumbbell in orbit. If the length of the dumbbell is assumed to be a constant, the equation may be obtained from the equations for an elastic dumbbell. Equation (38) and the form of Eq. (39) agree with those derived by Targoff.[10]

Targoff discusses stability for $\omega \geq 3\dot{\theta}$, where he shows that Eq. (39) can be approximated by the Mathieu equation. He concludes that the system is unstable in some cases but that a small amount of damping will stabilize the motion.

It is questionable whether sufficient damping will be present for small deflections. As will be demonstrated, including viscous damping in the equations for the cable does not produce linear terms in the equations for the transverse motion as it does in the equations for the axial motion and, hence, does not yield the desired damping effect. Flexural damping would certainly help, but the curvatures involved are so large for linear deflections of long thin cables that the flexural damping present would also be negligible.

Many approximations have been made in order to achieve a solution and the effects of these assumptions are difficult to assess. Even a numerical integration of the complete equations would be difficult due to the time-dependent boundary conditions. Furthermore, if complex structures are to be studied, there seems to exist no method of stability determination that can serve as a design procedure. However, a system such as the one discussed can be approximated by point masses and massless springs. The advantage offered is that the stability properties of the resulting equations of motion has been extensively studied.

Lumped-Mass Model

The continuous system in Fig. 1 can be approximated by a number of mass points connected by linear elastic massless springs as shown in Fig. 2. The Lagrangian for this model is now a function of the variables l_i, ϕ_i. The equations of motion are given by

$$d[\partial L/\partial \dot{q}_i]/dt - \partial L/\partial q_i = Q_i \qquad i = 1,2,\dots n+1$$

From the continuous analysis, we showed that certain terms from the kinetic energy contribution to the Lagrangian cancelled with the first term from the expansion of the gravity potential contribution when the center of mass moved in a Keplerian orbit. This fact will be demonstrated in the developments of the equations for the simple models that follow.

The kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{n+1} M_i\{\dot{R}^2 + (R\dot{\theta})^2 + \dot{l}_i^2 + (l_i\dot{\phi}_i)^2 + [2\dot{R}\dot{l}_i + 2R\dot{\theta}l_i\dot{\phi}_i]\cos(\phi_i - \theta) + \sin(\phi_i - \theta)[2R\dot{\theta}\dot{l} - 2\dot{R}l_i\dot{\phi}i]\}$$

The length of the spring between mass $i-1$ and i, from the law of cosines, is

$$e_i = [l_{i-1}^2 + l_i^2 - 2l_il_{i-1}\cos(\phi_{i-1} - \phi_i)]^{1/2}$$

If we define the unstretched length of each spring as l_{io}, the elastic potential may be written as

$$V_E = \frac{1}{2}\sum_{i=1}^{n+1} k_i[e_i - l_{io}]^2$$

where k is the spring constant of the ith spring. The gravity potential in expanded form is

$$V_G = -G\sum_{i=1}^{n+1}\left(\frac{M_i}{R}\right)\left[1 - \frac{l_i}{R}\cos(\phi_i - \theta) - \left(\frac{l_i^2}{2R^2}\right)[1 - 3\cos^2(\phi_i - \theta)] + 0\left(\frac{l_i^3}{R^3}\right)\right]$$

The Lagrangian is $L = T - V_E - V_G$ and the resulting equations of motion are

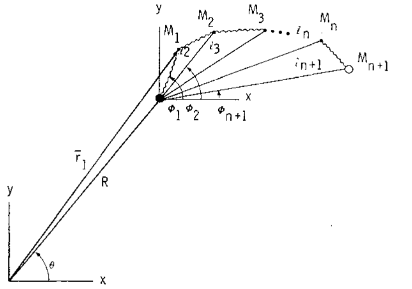

Fig. 2 Coordinate system for lumped mass model.

$$\ddot{l}_j = l_j\dot{\phi}_j^2 - (k_{j+1}/M_j)(1 - l_{(j+1)0}/e_{j+1}) \times [l_j - l_{j+1}\cos(\phi_j - \phi_{j+1})] - (k_j/M_j)(1 - l_{jo}/e_j)[l_j - l_{j-1}\cos(\phi_{j-1} - \phi_j)] + (Gl_j/2R^3)[1 + 3\cos 2(\phi_j - \theta)] \quad (41)$$

and

$$\ddot{\phi}_j = -2\dot{l}_j\dot{\phi}_j/l_j - (k_{j+1}/M_j)(1 - l_{(j+1)0}/e_j)[(l_{j+1}/l_j) \times \sin(\phi_j - \phi_{j+1})] + (k_j/M_j)(1 - l_{jo}/e_j)[(l_{j-1}/l_j) \times \sin(\phi_{j-1} - \phi_j)] - (3G/2R^3)\sin 2(\phi_j - \theta) \quad (42)$$

$$j = 1,2,\dots, n+1$$

The following definitions will account for the end conditions

$$l_0 = 0 \quad \text{and} \quad k_{n+2} = 0$$

These two definitions will yield the proper system of equations for the subsatellite-cable model considered here. The Lagrangian derived above will have the same form for any system of particles connected by linear elastic, massless springs. The only change will be the elastic potential. Its form will depend on the manner in which the system is tied together, but the method of derivation is unchanged.

To simulate damping which may be present an equivalent viscous damping term may be added to the equations. The damping force may be written as

$$F = C_i\dot{e}_i$$

where C_i is the damping coefficient of the ith spring and $\dot{e}_i$ is the rate of change of length of the spring. This adds a term to the right-hand side of Eqs. (41) and (42).

Numerical solutions of these equations have been obtained assuming a large end mass and two intermediate masses to represent the cable. A comparison of the first two natural frequencies of a continuous cable fixed between two walls with a two-mass model of the same system suggests that considerable insight can be gained from such a simple model.

To examine the effects of large deflections, the system of Eqs. (41) through (42) without damping was numerically integrated for varying tension and initial amplitude with

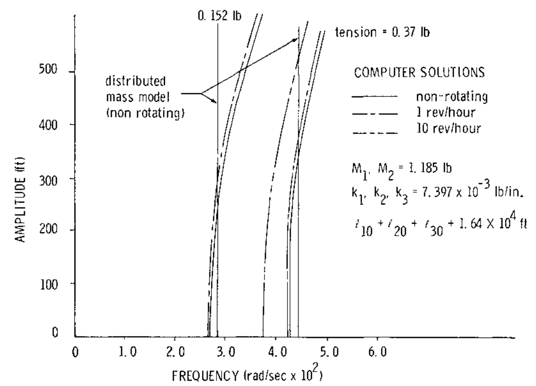

Fig. 3 Transverse amplitude vs frequency.

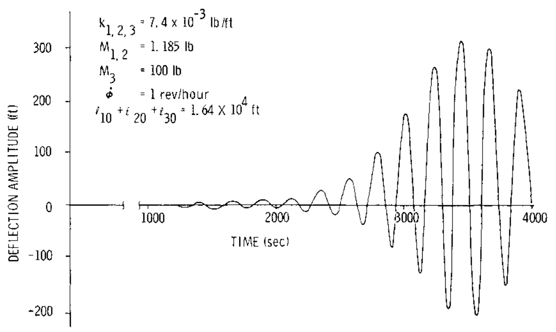

Fig. 4 Transverse amplitude vs time.

$R \to \infty$ (i.e., free space) and the end mass restrained at a constant radius but free in the tangential direction. Figure 3 shows amplitude vs frequency for two specific tensions and three rotational speeds. It shows a hardening spring effect as amplitude is increased and a reduction in natural frequency as rotational rate increases for constant tension.

When the end mass is unrestrained and the lowest transverse frequency is equal to or near to the axial frequency, a beating phenomenon can occur. When the axial mode is excited, the Coriolis effects directly exert a transverse force on each mass point of the cable. The acceleration of each mass point is different because the Coriolis acceleration is a function of the radial velocity and the spin rate. Hence the magnitude of the Coriolis acceleration varies along the cable. This causes transverse deflections to result due to the axial motion of the cable. It is this phenomenon which may cause large deflections in a spinning system.

The gravity forces on each mass point can be seen to be independent of its radius from the center of rotations for the linear approximation and, therefore, can not easily start transverse motions of the cable. To show this, we let $l \equiv l_s$, $\dot{l} \equiv 0$ and introduce this into the linearized form of Eq. (42) for $j = 1$. This yields an equation for the angular motion, ρ_i, which is independent of the steady-state length, namely

$$\ddot{\rho}_i = -(3G/R^3)\cos 2(\phi_o - \theta)\rho_i - (3G/2R^3)\sin 2(\phi_o - \theta) \quad (43)$$

This implies that an inextensible string, which was initially straight and spinning in the gravity gradient, would remain straight according to the linearized equations because the gravity gradient would cause all masses to speed up and slow down at the same rate independent of l_s. However, elastic stretching due to the gravity gradient and the resulting Coriolis acceleration cause oscillations and, as is the case in most elastic systems, the frequencies of the axial and transverse motions must not be in resonance or large deflections will result. Figure 4 shows a case where the transverse oscillations developed with time whereas Fig. 5 shows the beating phenomenon when the transverse oscillations were started initially.

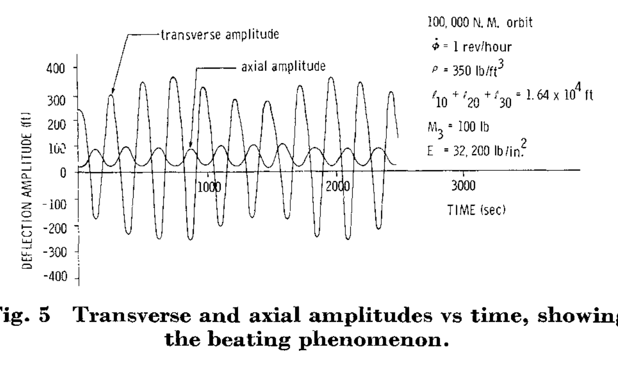

Fig. 5 Transverse and axial amplitudes vs time, showing the beating phenomenon.

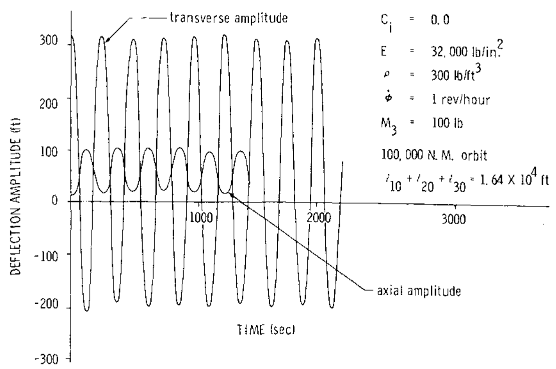

Fig. 6 Transverse and axial amplitudes vs time without damping.

Damping Effects

Damping is usually regarded as a "cure all" for most dynamics problems. Below are the linearized forms of the damping terms for a straight line configuration. They are

$$-(C_j/M_j)\dot{e}_j \cos\alpha_j - (C_{j+1}/M_j)\dot{e}_{j+1}\cos\epsilon_j \cong - (C_j/M_j)\{\dot{q}_{j-1}l_{(j-1)_o} + \dot{q}_j l_{j_o} - [\dot{q}_{j-1}l_{j_o} + \dot{q}_j l_{(j-1)_o}]\} - (C_{j+1}/M_j)\{\dot{q}_j l_{j_o} + \dot{q}_{j+1}l_{(j+1)_o} - [\dot{q}_j l_{(j+1)_o} + \dot{q}_{j+1}l_{j_o}]\} + 0(q)^2$$

and

$$-(C_j/M_j)\dot{e}_j \sin\alpha_j/l_j - (C_{j+1}/M_j)\dot{e}_{j+1}\sin\epsilon_j/l_j = 0 + 0(q)^2$$

where q_j's are the deviations in the lengths l_{i_o}. Note that the angular deviations do not enter into the linear damping terms.

This implies there is no direct damping in these coordinates for small deflections. This can be shown by comparing Fig. 6 with Fig. 7. The first case has no damping and the oscillations continue with a slight beat amplitude. In the second case, the axial oscillation is effectively damped in several oscillations and the transverse oscillations are reduced in amplitude initially, but continue to oscillate long after the axial motion has damped out. This indicates that damping cannot be relied upon to significantly reduce transverse line motion for small deflections (small in a mathematical sense and not necessarily small in a physical sense) as previous papers such as the one by Targoff have suggested.

Motion of the Radial Wire Out of the Orbital Plane

All the analysis to this point has allowed only motions in the plane of the orbit. This is certainly a large class of problems. However, it can be argued that a single mass on the end of a massless spring spinning in the orbital plane will experience no forces perpendicular to the orbital plane. The Coriolis, elastic, centrifugal, and gravity forces all act in the plane.

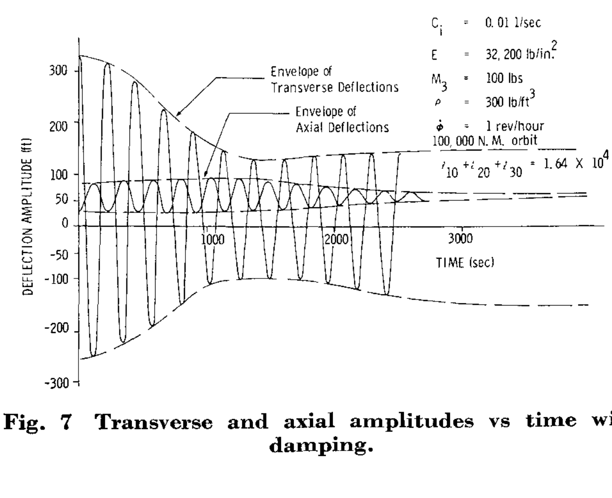

Fig. 7 Transverse and axial amplitudes vs time with damping.

Conclusions

For the distributed mass case, the complexity of the resulting nonlinear partial differential equations appears to preclude their use for motion and stability studies of complex systems. In particular, the equations must be linearized, and some of the linear coupling terms must be neglected in order to obtain solutions. On the other hand, if a system is approximated by lumped point masses connected by linear elastic massless springs, two practical advantages over the distributed mass approach are obtained: Lagrange's equations of motion may be systematically derived for any general system, and the resulting exact nonlinear ordinary differential equations may be integrated easily using a numerical integration routine. Thus, computer experiments can easily be conducted to obtain the motion of the system.

The particular results obtained by considering a lumped mass model of the spinning cable connected system of this study are as follows:

1) Very few masses are needed to predict accurately the lowest natural frequencies of the distributed mass cable; e.g., two masses will give the lowest frequency within $\sim 4\%$.

2) When viscous damping is included, the axial or stretching motion is effectively damped, but the transverse motion is damped only when deflections become so large as to be mathematically nonlinear: this seems to be a few percent of the length for the lowest mode.

3) A beating phenomenon occurs between the axial motion and the transverse when their frequencies are close together. Large deflections can also occur when these frequencies are near each other.

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